Work and energy when a dielectric is inserted into a capacitor

A parallel-plate capacitor whose capacitance C is 13.5 pF is charged by a battery to a potential difference $V = 12.5$ V between its plates. The charging battery is now disconnected, and a porcelain slab ($\kappa = 6.50$) is slipped between the plates.

(a) What is the potential energy of the capacitor before the slab is inserted?

KEY IDEA

We can relate the potential energy U_i of the capacitor to the capacitance C and either the potential V (with Eq. 25-22) or the charge q (with Eq. 25-21):

$$U_i = \tfrac{1}{2}CV^2 = \frac{q^2}{2C}.$$

Calculation: Because we are given the initial potential V (= 12.5 V), we use Eq. 25-22 to find the initial stored energy:

$$U_i = \tfrac{1}{2}CV^2 = \tfrac{1}{2}(13.5 \times 10^{-12}\text{ F})(12.5\text{ V})^2$$
$$= 1.055 \times 10^{-9}\text{ J} = 1055\text{ pJ} \approx 1100\text{ pJ}. \quad \text{(Answer)}$$

(b) What is the potential energy of the capacitor–slab device after the slab is inserted?

KEY IDEA

Because the battery has been disconnected, the charge on the capacitor cannot change when the dielectric is inserted. However, the potential *does* change.

Calculations: Thus, we must now use Eq. 25-21 to write the final potential energy U_f, but now that the slab is within the capacitor, the capacitance is κC. We then have

$$U_f = \frac{q^2}{2\kappa C} = \frac{U_i}{\kappa} = \frac{1055\text{ pJ}}{6.50}$$
$$= 162\text{ pJ} \approx 160\text{ pJ}. \quad \text{(Answer)}$$

When the slab is introduced, the potential energy decreases by a factor of κ.

The "missing" energy, in principle, would be apparent to the person who introduced the slab. The capacitor would exert a tiny tug on the slab and would do work on it, in amount

$$W = U_i - U_f = (1055 - 162)\text{ pJ} = 893\text{ pJ}.$$

If the slab were allowed to slide between the plates with no restraint and if there were no friction, the slab would oscillate back and forth between the plates with a (constant) mechanical energy of 893 pJ, and this system energy would transfer back and forth between kinetic energy of the moving slab and potential energy stored in the electric field.

Dielectrics: An Atomic View

What happens, in atomic and molecular terms, when we put a dielectric in an electric field? There are two possibilities, depending on the type of molecule:

1. *Polar dielectrics.* The molecules of some dielectrics, like water, have permanent electric dipole moments. In such materials (called *polar dielectrics*), the electric dipoles tend to line up with an external electric field as in Fig. 25-14. Because the molecules are continuously jostling each other as a result of their random thermal motion, this alignment is not complete, but it becomes more complete as the magnitude of the applied field is increased (or as the temperature, and thus the jostling, are decreased).The alignment of the electric dipoles produces an electric field that is directed opposite the applied field and is smaller in magnitude.

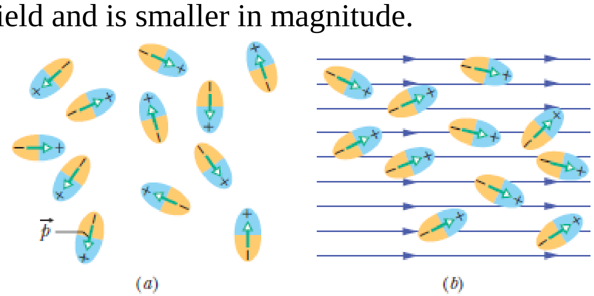
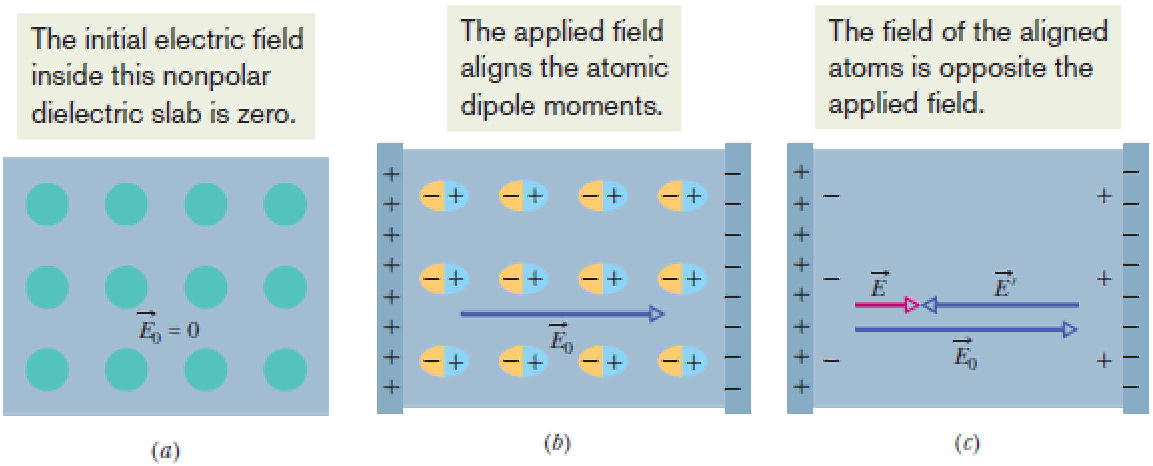


Fig. 25-14 (*a*) Molecules with a permanent electric dipole moment, showing their random orientation in the absence of an external electric field. (*b*) An electric field is applied, producing partial alignment of the dipoles. Thermal agitation prevents complete alignment.

2. *Nonpolar dielectrics.* Regardless of whether they have permanent electric dipole moments, molecules acquire dipole moments by induction when placed in an external electric field. In Section 24-8 (see Fig. 24-11), we saw that this occurs because the external field tends to "stretch" the molecules, slightly separating the centers of negative and positive charge.



Example 1 (Gauss' Law):

Relating the net enclosed charge and the net flux

Figure 23-7 shows five charged lumps of plastic and an electrically neutral coin. The cross section of a Gaussian surface S is indicated. What is the net electric flux through the surface if $q_1 = q_4 = +3.1$ nC, $q_2 = q_5 = -5.9$ nC, and $q_3 = -3.1$ nC?

KEY IDEA

The *net* flux Φ through the surface depends on the *net* charge q_{enc} enclosed by surface S.

Calculation: The coin does not contribute to Φ because it is neutral and thus contains equal amounts of positive and negative charge. We could include those equal amounts, but they would simply sum to be zero when we calculate the *net* charge enclosed by the surface. So, let's not bother. Charges q_4 and q_5 do not contribute because they are outside surface S. They certainly send electric field lines through the surface, but as much enters as leaves and no net flux is contributed. Thus, q_{enc} is only the sum $q_1 + q_2 + q_3$ and Eq. 23-6 gives us

$$\Phi = \frac{q_{enc}}{\varepsilon_0} = \frac{q_1 + q_2 + q_3}{\varepsilon_0}$$

$$= \frac{+3.1 \times 10^{-9}\text{ C} - 5.9 \times 10^{-9}\text{ C} - 3.1 \times 10^{-9}\text{ C}}{8.85 \times 10^{-12}\text{ C}^2/\text{N}\cdot\text{m}^2}$$

$$= -670\text{ N}\cdot\text{m}^2/\text{C}. \qquad \text{(Answer)}$$

The minus sign shows that the net flux through the surface is inward and thus that the net charge within the surface is negative.

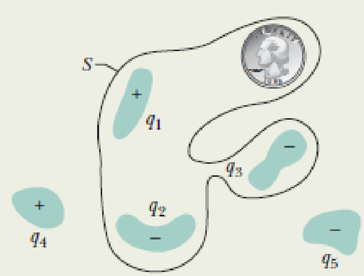


Fig. 23-7 Five plastic objects, each with an electric charge, and a coin, which has no net charge. A Gaussian surface, shown in cross section, encloses three of the plastic objects and the coin.

Example 22.10 **Charge on a hollow sphere**

A thin-walled, hollow sphere of radius 0.250 m has an unknown charge distributed uniformly over its surface. At a distance of 0.300 m from the center of the sphere, the electric field points radially inward and has magnitude 1.80×10^2 N/C. How much charge is on the sphere?

SOLUTION

IDENTIFY and SET UP: The charge distribution is spherically symmetric. As in Examples 22.5 and 22.9, it follows that the electric field is radial everywhere and its magnitude is a function only of the radial distance r from the center of the sphere. We use a spherical Gaussian surface that is concentric with the charge distribution and has radius $r = 0.300$ m. Our target variable is $Q_{encl} = q$.

EXECUTE: The charge distribution is the same as if the charge were on the surface of a 0.250-m-radius conducting sphere. Hence we can borrow the results of Example 22.5. We note that the electric field here is directed toward the sphere, so that q must be *negative*. Furthermore, the electric field is directed into the Gaussian surface, so that $E_\perp = -E$ and $\Phi_E = \oint E_\perp \, dA = -E(4\pi r^2)$.

By Gauss's law, the flux is equal to the charge q on the sphere (all of which is enclosed by the Gaussian surface) divided by ϵ_0. Solving for q, we find

$$q = -E(4\pi\epsilon_0 r^2) = -(1.80 \times 10^2 \text{ N/C})(4\pi) \times (8.854 \times 10^{-12} \text{ C}^2/\text{N} \cdot \text{m}^2)(0.300 \text{ m})^2 = -1.80 \times 10^{-9} \text{ C} = -1.80 \text{ nC}$$

EVALUATE: To determine the charge, we had to know the electric field at *all* points on the Gaussian surface so that we could calculate the flux integral. This was possible here because the charge distribution is highly symmetric. If the charge distribution is irregular or lacks symmetry, Gauss's law is not very useful for calculating the charge distribution from the field, or vice versa.

Example 2 (45 p625): Two charged concentric spherical shells have radii 10.0 cm and 15.0 cm. The charge on the inner shell is 4.00×10^{-8} C, and that on the outer shell is 2.00×10^{-8} C. Find the electric field (a) at r = 12.0 cm and (b) at r = 20.0 cm.

Example 3: The electric field at a distance of 0.145 m from the surface of a solid insulating sphere with radius 0.355 m is 1750 N/C. (a) Assuming the sphere's charge is uniformly distributed, what is the charge density inside it? (b) Calculate the electric field inside the sphere at a distance of 0.200 m from the center.

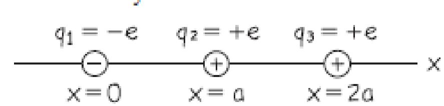
Potential Energy

Two point charges are located on the x-axis, $q_1 = -e$ at $x = 0$ and $q_2 = +e$ at $x = a$. (a) Find the work that must be done by an external force to bring a third point charge $q_3 = +e$ from infinity to $x = 2a$. (b) Find the total potential energy of the system of three charges.

SOLUTION

IDENTIFY and SET UP: Figure 23.10 shows the final arrangement of the three charges. In part (a) we need to find the work W that must be done on q_3 by an external force $\vec{F}_{ext}$ to bring q_3 in from infinity to $x = 2a$. We do this by using Eq. (23.10) to find the potential energy associated with q_3 in the presence of q_1 and q_2. In part (b) we use Eq. (23.11), the expression for the potential energy of a collection of point charges, to find the total potential energy of the system.

23.10 Our sketch of the situation after the third charge has been brought in from infinity.



EXECUTE: (a) The work W equals the difference between (i) the potential energy U associated with q_3 when it is at $x = 2a$ and (ii) the potential energy when it is infinitely far away. The second of these is zero, so the work required is equal to U. The distances between the charges are $r_{13} = 2a$ and $r_{23} = a$, so from Eq. (23.10),

$$W = U = \frac{q_3}{4\pi\epsilon_0}\left(\frac{q_1}{r_{13}} + \frac{q_2}{r_{23}}\right) = \frac{+e}{4\pi\epsilon_0}\left(\frac{-e}{2a} + \frac{+e}{a}\right) = \frac{+e^2}{8\pi\epsilon_0 a}$$

This is positive, just as we should expect. If we bring q_3 in from infinity along the $+x$-axis, it is attracted by q_1 but is repelled more strongly by q_2. Hence we must do positive work to push q_3 to the position at $x = 2a$.

(b) From Eq. (23.11), the total potential energy of the three-charge system is

$$U = \frac{1}{4\pi\epsilon_0}\sum_{i<j}\frac{q_iq_j}{r_{ij}} = \frac{1}{4\pi\epsilon_0}\left(\frac{q_1q_2}{r_{12}} + \frac{q_1q_3}{r_{13}} + \frac{q_2q_3}{r_{23}}\right)$$
$$= \frac{1}{4\pi\epsilon_0}\left[\frac{(-e)(e)}{a} + \frac{(-e)(e)}{2a} + \frac{(e)(e)}{a}\right] = \frac{-e^2}{8\pi\epsilon_0 a}$$

EVALUATE: Our negative result in part (b) means that the system has lower potential energy than it would if the three charges were infinitely far apart. An external force would have to do *negative* work to bring the three charges from infinity to assemble this entire arrangement and would have to do *positive* work to move the three charges back to infinity.

Example 23.4 Potential due to two point charges

An electric dipole consists of point charges $q_1 = +12$ nC and $q_2 = -12$ nC placed 10.0 cm apart (Fig. 23.13). Compute the electric potentials at points a, b, and c.

SOLUTION

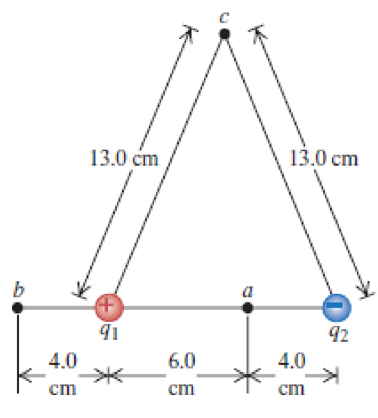
IDENTIFY and SET UP: This is the same arrangement as in Example 21.8, in which we calculated the electric *field* at each point by doing a *vector* sum. Here our target variable is the electric *potential* V at three points, which we find by doing the *algebraic* sum in Eq. (23.15).

EXECUTE: At point a we have $r_1 = 0.060$ m and $r_2 = 0.040$ m, so Eq. (23.15) becomes

$$V_a = \frac{1}{4\pi\epsilon_0}\sum_i\frac{q_i}{r_i} = \frac{1}{4\pi\epsilon_0}\frac{q_1}{r_1} + \frac{1}{4\pi\epsilon_0}\frac{q_2}{r_2}$$
$$= (9.0\times10^9\ \text{N}\cdot\text{m}^2/\text{C}^2)\frac{12\times10^{-9}\ \text{C}}{0.060\ \text{m}}$$
$$+ (9.0\times10^9\ \text{N}\cdot\text{m}^2/\text{C}^2)\frac{(-12\times10^{-9}\ \text{C})}{0.040\ \text{m}}$$
$$= 1800\ \text{N}\cdot\text{m/C} + (-2700\ \text{N}\cdot\text{m/C})$$
$$= 1800\ \text{V} + (-2700\ \text{V}) = -900\ \text{V}$$

In a similar way you can show that the potential at point b (where $r_1 = 0.040$ m and $r_2 = 0.140$ m) is $V_b = 1930$ V and that the potential at point c (where $r_1 = r_2 = 0.130$ m) is $V_c = 0$.

23.13 What are the potentials at points a, b, and c due to this electric dipole?



is closer to the +12-nC charge than the −12-nC charge. Finally, point c is equidistant from the +12-nC charge and the −12-nC charge, so the potential there is zero. (The potential is also equal to zero at a point infinitely far from both charges.)

Comparing this example with Example 21.8 shows that it's much easier to calculate electric potential (a scalar) than electric field (a vector). We'll take advantage of this simplification whenever possible.

Example 23.5 Potential and potential energy

Compute the potential energy associated with a +4.0-nC point charge if it is placed at points *a*, *b*, and *c* in Fig. 23.13.

SOLUTION

IDENTIFY and SET UP: The potential energy U associated with a point charge q at a location where the electric potential is V is $U = qV$. We use the values of V from Example 23.4.

EXECUTE: At the three points we find

$$U_a = qV_a = (4.0 \times 10^{-9}\text{ C})(-900\text{ J/C}) = -3.6 \times 10^{-6}\text{ J}$$
$$U_b = qV_b = (4.0 \times 10^{-9}\text{ C})(1930\text{ J/C}) = 7.7 \times 10^{-6}\text{ J}$$
$$U_c = qV_c = 0$$

All of these values correspond to U and V being zero at infinity.

EVALUATE: Note that *zero* net work is done on the 4.0-nC charge if it moves from point c to infinity *by any path.* In particular, let the path be along the perpendicular bisector of the line joining the other two charges q_1 and q_2 in Fig. 23.13. As shown in Example 21.8 (Section 21.5), at points on the bisector, the direction of $\vec{E}$ is perpendicular to the bisector. Hence the force on the 4.0-nC charge is perpendicular to the path, and no work is done in any displacement along it.

Find the equivalent capacitance of the five-capacitor network shown in Fig. 24.10a.

SOLUTION

IDENTIFY and SET UP: These capacitors are neither all in series nor all in parallel. We can, however, identify portions of the arrangement that *are* either in series or parallel. We combine these as described in Problem-Solving Strategy 24.1 to find the net equivalent capacitance, using Eq. (24.5) for series connections and Eq. (24.7) for parallel connections.

EXECUTE: The caption of Fig. 24.10 outlines our procedure. We first use Eq. (24.5) to replace the 12-μF and 6-μF series combination by its equivalent capacitance C':

$$\frac{1}{C'} = \frac{1}{12\ \mu\text{F}} + \frac{1}{6\ \mu\text{F}} \qquad C' = 4\ \mu\text{F}$$

This gives us the equivalent combination of Fig. 24.10b. Now we see three capacitors in parallel, and we use Eq. (24.7) to replace them with their equivalent capacitance C'':

$$C'' = 3\ \mu\text{F} + 11\ \mu\text{F} + 4\ \mu\text{F} = 18\ \mu\text{F}$$

This gives us the equivalent combination of Fig. 24.10c, which has two capacitors in series. We use Eq. (24.5) to replace them with their equivalent capacitance C_{eq}, which is our target variable (Fig. 24.10d):

$$\frac{1}{C_{eq}} = \frac{1}{18\ \mu\text{F}} + \frac{1}{9\ \mu\text{F}} \qquad C_{eq} = 6\ \mu\text{F}$$

EVALUATE: If the potential difference across the entire network in Fig. 24.10a is $V_{ab} = 9.0$ V, the net charge on the network is $Q = C_{eq}V_{ab} = (6\ \mu\text{F})(9.0\text{ V}) = 54\ \mu\text{C}$. Can you find the charge on, and the voltage across, each of the five individual capacitors?

24.10 (a) A capacitor network between points a and b. (b) The 12-μF and 6-μF capacitors in series in (a) are replaced by an equivalent 4-μF capacitor. (c) The 3-μF, 11-μF, and 4-μF capacitors in parallel in (b) are replaced by an equivalent 18-μF capacitor. (d) Finally, the 18-μF and 9-μF capacitors in series in (c) are replaced by an equivalent 6-μF capacitor.